Continuous Approximate Synthesis of Planar Function-generators Minimising The Design Error

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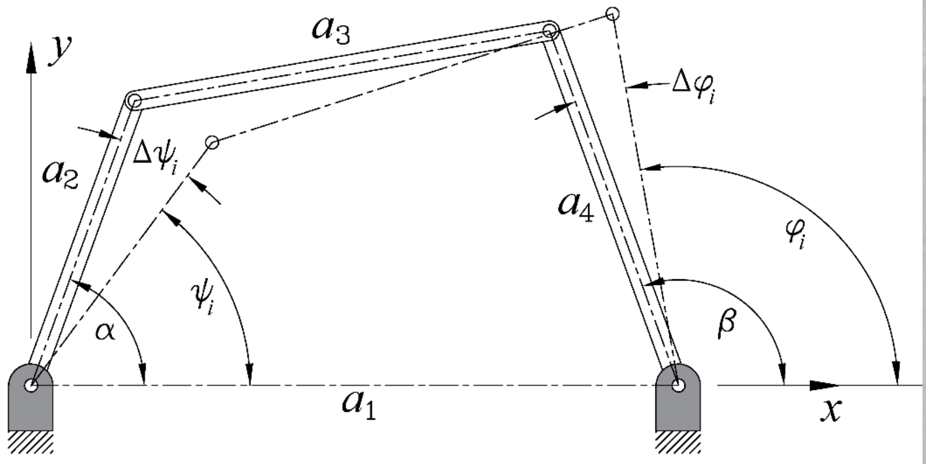
Introduction

- It has been observed that as the cardinality of the prescribed input-output (I/O) data set increases, the linkages that minimise the Euclidean norms of the design and structural errors tend to converge to the same linkage.
- This implies that minimising the structural error can be accomplished implicitly by minimising the design error.
- We propose to allow the cardinality of the data set to be infinite, thereby reposing the discrete approximate synthesis problem as a continuous one.
- This will be accomplished by integrating the synthesis equation in the range between minimum and maximum I/O pairs.
- In this work we prove that the lower bound on the Euclidean norm of the design error for a planar 4R function-generating linkage exists and is attained with continuous approximate synthesis.

Function Generation

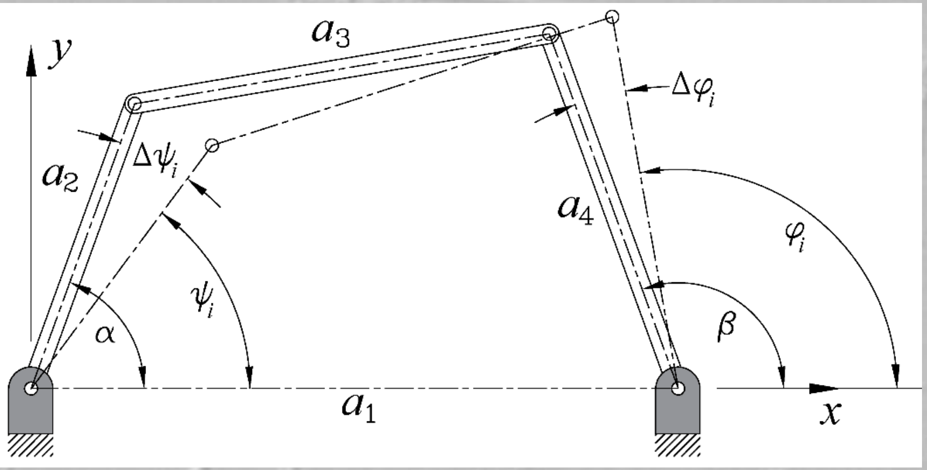
- A 4R planar function-generator consists of four rigid bars, or links, joined by four revolute joints where one of the bars is relatively non-moving.
- The link lengths can be chosen such that the output angle of the follower is a function of the input angle of the driver.



- In general, exact algebraic dimensional synthesis for function generation requires three I/O pairs.
- Approximate algebraic dimensional synthesis requires $m > 3$ I/O pairs.
- The more I/O pairs leads to a more precise function generator.
- Hence, we propose continuous approximate synthesis by allowing $m = \infty$.

Discrete Design Error Minimisation

- We consider the optimisation problem to be the approximate solution to an overdetermined system of linear equations with the minimum error.
- We use the Freudenstein synthesis equations.
- The i^{th} configuration of the mechanism is governed by:



$$k_1 + k_2\cos(\varphi_i) - k_3\cos(\psi_i) = \cos(\psi_i - \varphi_i).$$

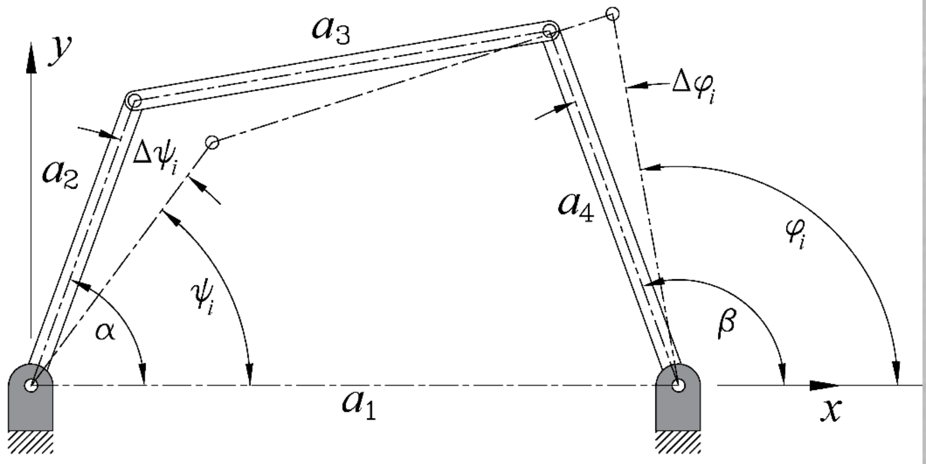
- Where ψ_i and φ_i are the i^{th} I/O variables, respectively, and k_1, k_2, and k_3 are the Freudenstein Parameters.

Discrete Design Error Minimisation

- The k's are the link length ratios

$$k_1 = \frac{a_1^2 + a_2^2 + a_4^2 - a_3^2}{2a_2a_4};$$

$$k_2 = \frac{a_1}{a_2}; \quad k_3 = \frac{a_1}{a_4}.$$

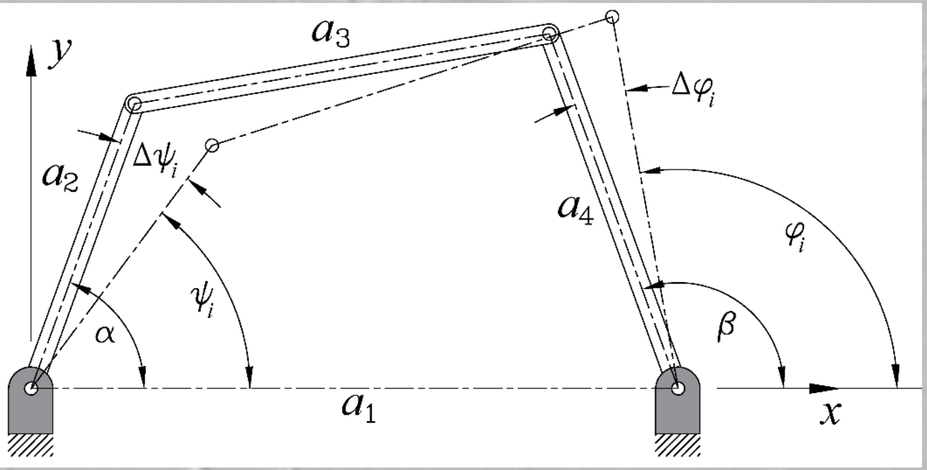


- Given k_1, k_2, and k_3 the corresponding link lengths, scaled by a_1, are:

$$a_1 = 1; \quad a_2 = \frac{1}{k_2}; \quad a_4 = \frac{1}{k_3}; \quad a_3 = (a_1^2 + a_2^2 + a_4^2 - 2a_2a_4k_1)^{1/2}.$$

Discrete Design Error Minimisation

- The set of I/O Freudenstein equations can be expressed in vector-matrix form:



$$\mathbf{Sk} = \mathbf{b},$$

- where **S** is the m x 3 synthesis matrix whose i^{th} row is the 1 x 3 array s_i, **b** is an m-dimensional vector, and **k** is 3-dimensional vector of the Freudenstein parameters (design variables).
- For the planar 4R function-generator we have:

$$s_i = \begin{bmatrix} 1 & \cos\phi_i & -\cos\psi_i \end{bmatrix}; \quad b_i = \begin{bmatrix} \cos(\psi_i - \phi_i) \end{bmatrix}; \quad \mathbf{k} = \begin{bmatrix} k_1 & k_2 & k_3 \end{bmatrix}^T.$$

Discrete Design Error Minimisation

- The synthesised linkage will only be capable of generating the desired function approximately.
- The design error is the algebraic difference of the left-hand side of the Freudenstein equations less the right-hand side.
- Because we will be comparing errors associated with different cardinalities, we now include the cardinality *m* in the definition.
- The *m*-dimensional design error vector $\mathbf{d}_m$ for a set of *m* (*m* > 3) I/O pairs, $\{(\psi_i, \phi_i) i = 1 \ldots m\}$, is defined as:

$$\mathbf{d}_m = \mathbf{S}_m \mathbf{k} - \mathbf{b}_m .$$

- If the outputs prescribed by the functional relationship correspond precisely to the outputs generated by the mechanism, then $\|\mathbf{d}_m\| = 0.$
- However, for a general prescribed function $\phi_{pres}(\psi)$, $\|\mathbf{d}_m\| \neq 0.$

Discrete Design Error Minimisation

- We seek the Freudenstein parameter vector that minimises the norm of the design error vector.
- The optimal Freudenstein parameters $\mathbf{k}_m^*$ for this norm are:

$$\mathbf{k}_m^* = \mathbf{S}_m^+ \mathbf{b}_m,$$

where $\mathbf{S}_m^+$ is the Moore-Penrose generalized inverse of the synthesis matrix.

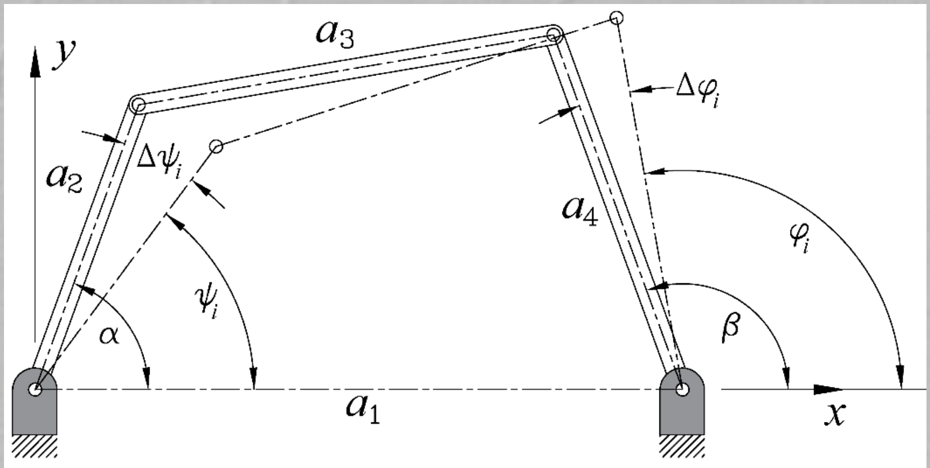
- The corresponding minimal design error is:

$$\min_{\mathbf{k}} \left\| \mathbf{d}_m \right\|_2 = \left\| \mathbf{d}_m^* \right\|_2 = \left\| \mathbf{I}_m - \mathbf{S}_m \mathbf{S}_m^+ \mathbf{b}_m \right\|_2.$$

Discrete Design Error Minimisation

- For numerical stability it is desirable to have a well-conditioned synthesis matrix otherwise numerical values of $\mathbf{S}_m^+$ may be significantly distorted by very small singular values leading to optimised **k** that imply a mechanism that poorly approximates the desired function.
- Hence dial zeros α and β are introduced to maximise the inverse of the condition number, 1/κ, i.e. the ratio of the smallest to the largest singular values of the synthesis matrix:

$$\psi = \alpha + \Delta\psi; \quad \phi = \beta + \Delta\phi.$$



Discrete Design Error Minimisation

- Including the dial zeros, the synthesis equation becomes:

$$k_1 + k_2\cos(\beta + \Delta\phi) - k_3\cos(\alpha + \Delta\psi) = \cos(\alpha + \Delta\psi - \beta - \Delta\phi).$$

- Now the I\O pairs are a set of incremental changes in angle:

$$\{(\Delta\psi_i, \Delta\phi_i) i = 0 \dots m\}.$$

- $\mathbf{d}_m^*$, $\mathbf{k}_m^*$, and $\mathbf{S}_m^*$ are now also functions of the dial zeros.
- The design error minimisation problem can now be efficiently solved in a least squares sense in two steps:

1. Find the dial zeros to maximise the inverse of the condition number $1/\kappa_m(\alpha, \beta)$ of the synthesis matrix $\mathbf{S}_m$;
2. Find the corresponding optimal Freudenstein parameters using $\mathbf{k}_m^* = \mathbf{S}_m^+ \mathbf{b}_m$.

Continuous Design Error Minimisation

- A major issue associated with the discrete approach to the design error minimisation is the appropriate choice for the cardinality of the I/O pair data set such that the minimisation of the structural error is implied.
- Indeed, the choice of *m* depends on the prescribed function $\Delta\phi(\Delta\psi)$ and *m* is generally fixed when some level of convergence is observed.
- For the example used in an earlier paper, *m* = 40 was observed to be a good choice.
- We now propose to evaluate the design error over the continuous range $[\Delta\psi_{\min}, \Delta\psi_{\max}]$ of the prescribed function.
- This requires the function to be continuous over the range of the function.
- It also requires a different vector space, denoted $\wp^{\circ}([\Delta\psi_{\min}, \Delta\psi_{\max}])$, where upon the following 2-norm has been imposed:

$$\forall f \in \wp^{\circ}([\Delta\psi_{\min}, \Delta\psi_{\max}]),\ \|f\|_2 = \left(\int_{\Delta\psi_{\min}}^{\Delta\psi_{\max}} |f|^2(\psi)\, d\psi \right)^{1/2}.$$

Continuous Design Error Minimisation

- Assuming that the prescribed function belongs to $\wp^{\circ}([\Delta\psi_{\min}, \Delta\psi_{\max}])$, the design error is:

$$\left\|\mathbf{d}(\alpha,\beta)\right\|_2 = \left(\int_{\Delta\psi_{\min}}^{\Delta\psi_{\max}} (k_1 + k_2\cos(\beta+\Delta\phi) - k_3\cos(\alpha+\Delta\psi) - \cos(\alpha+\Delta\psi-\beta-\Delta\phi))^2 \, d\Delta\psi\right)^{1/2}.$$

- After some algebraic manipulation it can be shown that the above equation is a quadratic function in terms of the Freudenstein parameters:

$$\left\|\mathbf{d}(\alpha,\beta)\right\|_2 = \mathbf{k}^T\mathbf{A}(\alpha,\beta)\mathbf{k} - 2\mathbf{e}(\alpha,\beta)^T\mathbf{k} + c(\alpha,\beta).$$

- $\mathbf{A}(\alpha, \beta)$ is a 3 x 3 symmetric matrix whose six distinct elements a_{ij} are:

Continuous Design Error Minimisation

$$a_{11} = \int_{\Delta\psi_{\min}}^{\Delta\psi_{\max}} d\Delta\psi;$$

$$a_{12} = \cos(\beta)\int_{\Delta\psi_{\min}}^{\Delta\psi_{\max}} \cos(\Delta\phi)d\Delta\psi - \sin(\beta)\int_{\Delta\psi_{\min}}^{\Delta\psi_{\max}} \sin(\Delta\phi)d\Delta\psi;$$

$$a_{13} = -\cos(\alpha)\int_{\Delta\psi_{\min}}^{\Delta\psi_{\max}} \cos(\Delta\psi)d\Delta\psi + \sin(\alpha)\int_{\Delta\psi_{\min}}^{\Delta\psi_{\max}} \sin(\Delta\psi)d\Delta\psi;$$

$$a_{22} = \cos(\beta)^2\int_{\Delta\psi_{\min}}^{\Delta\psi_{\max}} \cos(\Delta\phi)^2 d\Delta\psi - 2\cos(\beta)\sin(\beta)\int_{\Delta\psi_{\min}}^{\Delta\psi_{\max}} \cos(\Delta\phi)\sin(\Delta\phi)d\Delta\psi$$
$$+\sin(\beta)^2\int_{\Delta\psi_{\min}}^{\Delta\psi_{\max}} \sin(\Delta\phi)^2 d\Delta\psi;$$

$$a_{23} = -\cos(\alpha)\cos(\beta)\int_{\Delta\psi_{\min}}^{\Delta\psi_{\max}} \cos(\Delta\psi)\cos(\Delta\phi)d\Delta\psi + \cos(\alpha)\sin(\beta)\int_{\Delta\psi_{\min}}^{\Delta\psi_{\max}} \cos(\Delta\psi)\sin(\Delta\phi)d\Delta\psi$$
$$+\sin(\alpha)\cos(\beta)\int_{\Delta\psi_{\min}}^{\Delta\psi_{\max}} \sin(\Delta\psi)\cos(\Delta\phi)d\Delta\psi - \sin(\alpha)\sin(\beta)\int_{\Delta\psi_{\min}}^{\Delta\psi_{\max}} \sin(\Delta\psi)\sin(\Delta\phi)d\Delta\psi;$$

$$a_{33} = \cos(\alpha)^2\int_{\Delta\psi_{\min}}^{\Delta\psi_{\max}} \cos(\Delta\psi)^2 d\Delta\psi - 2\cos(\alpha)\sin(\beta)\int_{\Delta\psi_{\min}}^{\Delta\psi_{\max}} \cos(\Delta\psi)\sin(\Delta\psi)d\Delta\psi$$
$$+\sin(\alpha)^2\int_{\Delta\psi_{\min}}^{\Delta\psi_{\max}} \sin(\Delta\psi)^2 d\Delta\psi.$$

Continuous Design Error Minimisation

- $\mathbf{e}(\alpha, \beta)$ is a 3-dimensional vector whose three elements are:

$$e_1 = \int_{\Delta\psi_{\min}}^{\Delta\psi_{\max}} \cos(\alpha + \Delta\psi - \beta - \Delta\phi) d\Delta\psi;$$

$$e_2 = \int_{\Delta\psi_{\min}}^{\Delta\psi_{\max}} \left(\cos(\beta + \Delta\phi)\cos(\alpha + \Delta\psi - \beta - \Delta\phi)\right) d\Delta\psi;$$

$$e_3 = -\int_{\Delta\psi_{\min}}^{\Delta\psi_{\max}} \left(\cos(\alpha + \Delta\psi)\cos(\alpha + \Delta\psi - \beta - \Delta\phi)\right) d\Delta\psi.$$

- And finally $c(\alpha, \beta)$ is a scalar having the form:

$$c = \int_{\Delta\psi_{\min}}^{\Delta\psi_{\max}} \cos(\alpha + \Delta\psi - \beta - \Delta\phi)^2 d\Delta\psi.$$

Continuous Design Error Minimisation

- If $\mathbf{A}(\alpha, \beta)$ is positive definite, the optimal Freudenstein parameters $\mathbf{k}^*(\alpha, \beta)$ which minimise $\|\mathbf{d}(\alpha, \beta)\|_2$ are:

$$\mathbf{k}^*(\alpha, \beta) = \mathbf{A}^{-1}(\alpha, \beta)\mathbf{e}(\alpha, \beta).$$

- The minimal design error is:

$$\min_{\mathbf{k}} \|\mathbf{d}(\alpha, \beta)\|_2 = \min_{\mathbf{k}} \|\mathbf{d}^*(\alpha, \beta)\|_2 = c(\alpha, \beta) - \mathbf{e}(\alpha, \beta)^T \mathbf{A}^{-1}(\alpha, \beta)\mathbf{e}(\alpha, \beta).$$

- As for the discrete approach, the design error minimisation problem can be solved in two steps:

1. Find the dial zeros to maximise the inverse of the condition number $1/\kappa_m(\alpha, \beta)$ of the synthesis matrix $\mathbf{A}(\alpha, \beta)$;
2. Find the corresponding optimal Freudenstein parameters using the first equation above.

Continuous Design Error Minimisation

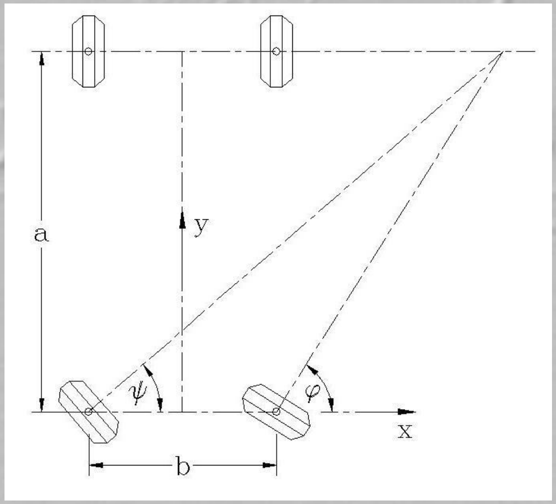
- Perhaps the main result of this paper is to be found in Proposition 5, which is proved in the paper, but is to complicated to summarize here, is the following:

$$\lim_{m \to \inf} \frac{1}{\kappa_m}(\alpha_m, \beta_m) = \frac{1}{\kappa^*}(\alpha^*, \beta^*).$$

Example

- The preceding results for continuous approximate synthesis that minimises the design error are now illustrated with an example.
- Let the prescribed function be the Ackerman steering condition for terrestrial vehicles.
- The steering condition can be expressed as a trigonometric function whose variables are illustrated in the figure.

$$\sin(\Delta\phi_{pres} - \Delta\psi) - (b/a)\sin(\Delta\psi)\sin(\Delta\phi_{pres}) = 0.$$



Example

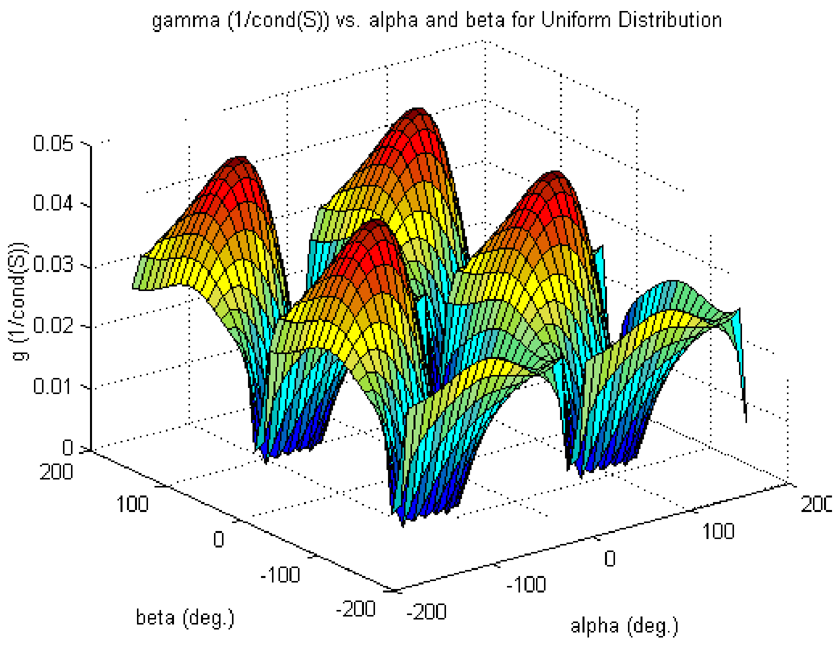
- With the dial zeros, the expression for the steering condition becomes:

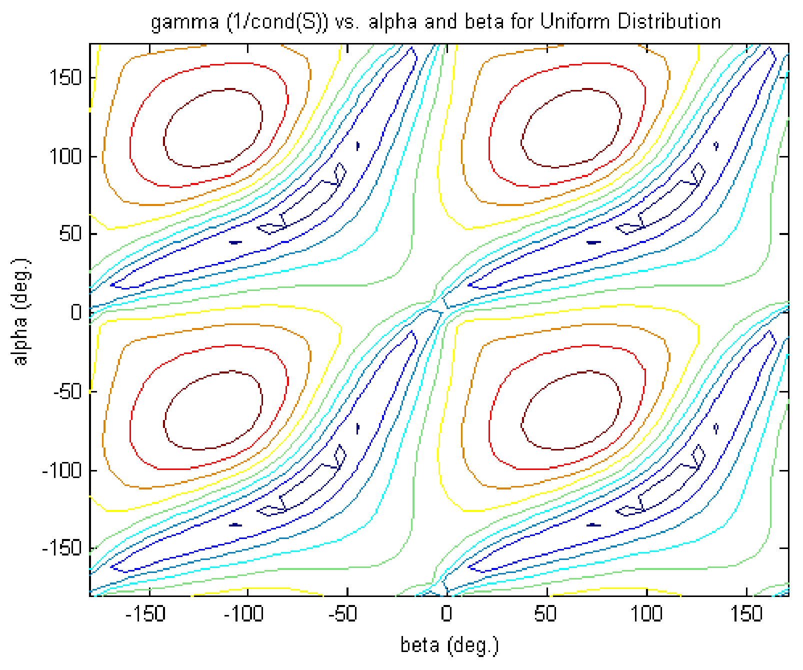
$$\sin(\beta + \Delta\phi_{pres} - \alpha - \Delta\psi) - (b/a)\sin(\alpha + \Delta\psi)\sin(\beta + \Delta\phi_{pres}) = 0.$$

- For our example *b*/*a*=0.5 and $[\Delta\psi_{\min}, \Delta\psi_{\max}] = [-40.00, 30.00]$, where angles are specified in degrees.
- With these values, the prescribed function, i.e. the steering condition, is continuously differentiable. Hence, Proposition 5 must apply.
- The multi-dimensional Nelder-Mead downhill simplex algorithm is employed to find the optimal values for the dial zeros.
- In Matlab, fminsearch finds the minimum of a scalar function of several variables, starting at an initial estimate. This is generally referred to as unconstrained nonlinear optimization.

Example

- The initial estimate is determined by plotting $1/\kappa$ over a range of values for α and β, illustrated in the figure:





Example

- The table below lists (α_m^*, β_m^*) for different values of m, as well as (α^*, β^*)

m	α_m^*	β_m^*	α^*	β^*
10	-61.80	67.320	-	-
40	-62.17	68.73	-	-
100	-62.23	69.03	-	-
400	-62.26	69.17	-	-
1000	-62.27	69.20	-	-
∞	-	-	-62.27	69.22

Example

- From the optimal dial zeros, it is now possible to compute the optimal Freudenstein parameters.
- The design error norms have been normalized by dividing by $\sqrt{m}$ for comparison for different values of *m* as well as the values using the continuous approach.

m	k_1	k_2	k_3	κ_m	κ^*	$\|\mathbf{d}_m\|_2$	$\|\mathbf{d}^*\|_2$
10	-0.993	0.412	-0.429	18.24	-	6.93×10^{-4}	-
40	-1.001	0.406	-0.425	20.79	-	6.44×10^{-4}	-
100	-1.003	0.405	-0.424	21.38	-	6.31×10^{-4}	-
400	-1.003	0.404	-0.424	21.69	-	6.24×10^{-4}	-
1000	-1.004	0.404	-0.424	21.75	-	6.23×10^{-4}	-
∞	-1.004	0.404	-0.424	-	475.03	-	6.23×10^{-4}

Conclusions

- The example presented employed the Matlab function quadl, which employs recursive adaptive Lobatto quadrature.
- Continuous approximate synthesis eliminates the problem of determining an appropriate cardinality for the data-set.
- Evaluating the design error over the whole range of the function requires the use of a functional normed space, thereby changing the discrete approximate synthesis problem to a continuous approximate synthesis problem.
- Assuming that the prescribed function is continuously differentiable, it was shown that the dial zeros, the optimal Freudenstein parameters, and the minimal design error for discrete approximate synthesis converge towards the dial zeros, the optimal Freudenstein parameters and the minimal design error for continuous approximate synthesis.
- In other words, the continuous approach corresponds to the discrete approach after setting the cardinality of the I/O set to $m = \infty$.

Conclusions

- The extension of this work is to investigate how the structural error bounds the design error.
- First, it should be determined if the structural error minimisation problem can be formulated and, more importantly solved, using the continuous approach.
- Second, it should be investigated whether in this case too, the continuous approach corresponds to the discrete approach with $m = \infty$.
- This is certainly much more challenging due to increased complexity of the continuous structural error minimisation problem, which is a non-linear problem with equality constraints, compared to the continuous design error minimisation problem, which is a quadratic problem without any constraints.
- Finally, one might ask whether our developments could be applied to other mechanism topologies, such as planar mechanisms possessing prismatic joints, as well as spherical, or spatial linkages.

Engineering Humor

- Why do engineers confuse Halloween with Christmas Day?
- Because oct 31 = dec 25!